A Stationarity Principle for the Eigenvalue Problem for Rotating Structures

Leonard Meirovitch*
Virginia Polytechnic Institute and State University, Blacksburg, Va.

An important question associated with the eigenvalue problem for flexible gyroscopic systems is that of discretization of continuous elastic members. If discretization is performed by the assumed modes method, the question arises as to the type of functions to be used in series expansions. In particular, the question is whether one should use rotating-appendage eigenfunctions, fixed-base eigenfunctions, or any other set of admissible functions. The answer to this question is provided by a stationarity principle for rotating structures developed and proved in this paper. On the basis of this principle, it can be concluded that discretization by means of admissible functions is quite sufficient and the use of appendage eigenfunctions is unnecessary, provided the set of admissible functions is complete. The principle has important implications, not only in a modal analysis for the response, but also in a stability analysis of flexible spacecraft.

Introduction

A PROBLEM of current interest is that of dynamic characteristics of spinning flexible bodies. The motion of such bodies can be described by the rotation of a given reference frame and the elastic displacement of the body relative to that frame. The associated mathematical formulation is in terms of both ordinary and partial differential equations; such a system has come to be known as hybrid. Our concern is with the case in which the body undergoes small rotational and elastic displacements from the uniform-spin equilibrium state, so that in essence we are concerned with linear gyroscopic systems.

A common procedure for analyzing hybrid systems is discretization, whereby the partial differential equations are replaced by ordinary ones. There are many discretization procedures in common use, such as the Rayleigh-Ritz method, the finite-element method, the lumped parameter method, etc. The Rayleigh-Ritz method consists of expanding the displacement of a continuous elastic member in a finite series of known space-dependent functions multiplied by time-dependent generalized coordinates. Because the space-dependent functions used in such expansions generally represent vibration modes of a closely related system, the Rayleigh-Ritz method often is referred to as the assumed modes method.† The finite-element method can be regarded as a localized Rayleigh-Ritz method, whereas the lumped parameter method simply implies spatial discretization. All three methods are discussed in Ref. 1. When the series expansion is used merely for discretization purposes in conjunction with a stability analysis, rather than for a computation of the system natural modes of vibration, the term "assumed modes method" is more appropriate than the "Rayleigh-Ritz method."

The Rayleigh-Ritz method has been used extensively to obtain estimates of natural frequencies and modes of vibration of natural (nongyroscopic) distributed-parameter systems. The method is based on the stationarity property of Rayleigh's quotient, in which we note that for nonrotating structures Rayleigh's quotient can be readily defined. This paper proves that a stationarity property exists also for gyroscopic systems. To this end, it introduces a definition of Rayleigh's quotient based on the eigenvalue problem formulation presented by this author in an earlier paper.[2]

A question that arises in the discretization of a system by means of the assumed modes method is the nature of the functions to be used in series expansions. The choice is between comparison functions and admissible functions, where the first satisfy all the boundary conditions associated with a given elastic member and the second, only the geometric boundary conditions. (More detailed definitions of these classes of functions are given later in the paper.) In the case of flexible gyroscopic systems there is another factor to consider, namely, the effect, if any, of the system nominal uniform rotation on the choice of functions. There appears to be widespread belief that superior results can be obtained if discretization of a given appendage is achieved by means of the eigenfunctions of the uniformly rotating appendage. This idea has considerable intuitive appeal and is responsible for renewed interest in the solution of the eigenvalue problem of uniformly rotating elastic distributed members, such as strings, rods, membranes, and plates. This paper demonstrates, however, that there is no particular advantage in using such eigenfunctions and, indeed, comparison functions and admissible functions can yield equally good results. This conclusion has far reaching implications in the treatment of spinning flexible bodies, since it eliminates the apparent need to solve eigenvalue problems associated with rotating appendages. Such solutions are not only difficult to obtain at times, but they solve no exact problem, because, in defining the eigenvalue problem for the elastic perturbations, the angular perturbations are ignored, although they often are of the same order of magnitude as the elastic perturbations. A survey of pertinent literature on the subject should help clarify many of these points.

In one of the earliest papers concerned with the attitude stability of spinning spacecraft with flexible appendages, Thomson and Reiter[3] used the assumed modes method in conjunction with an expansion in terms of the eigenfunctions of the fixed-base beam to discretize the system. Note that these eigenfunctions are merely comparison functions for uniformly rotating beams, since they do not satisfy the differential equation of the eigenvalue problem. The assumed modes method in conjunction with the same type of com-

Received Nov. 5, 1975; presented as Paper 76-184 at the AIAA 14th Aerospace Sciences Meeting, Washington, D. C. Jan. 26-28, 1976; revision received March 22, 1976. The numerical results were obtained by means of a computer program written by A. L. Hale.

Index categories: Spacecraft Attitude Dynamics and Control; Structural Dynamic Analysis.

*Professor of Engineering Science and Mechanics. Associate Fellow AIAA.

†Some investigators refer to this method as "modal analysis." Because modal analysis generally implies a method that uses the system natural modes to transform a partial differential equation into a set of independent ordinary differential equations, the use of the term for a mere discretization procedure is not appropriate.

parison functions has been used by various other investigators[4-8] for stability analyses and/or for the calculation of the system natural frequencies. Reference 4 presents, in addition, qualitative comments on the effect of using eigenfunctions of uniformly rotating elastic members on the system natural frequencies. It should be pointed out that the authors of Refs. 3-8 did not actually use terminology such as comparison functions and admissible functions. The use of this terminology to discuss Refs. 3-8 was deemed appropriate here, because it permitted placement of these references in proper perspective relative to the present paper.

The practice of using eigenfunctions of the fixed-base cantilever beams to represent the elastic motion of rotating flexible appendages has come under severe criticism by Rakowski and Renard.[9] Indeed, the major conclusion of Ref. 9 is: "Significant errors are introduced in studies of nutational behavior by using standard cantilever modes and frequencies rather than those of the rotating structure." This conclusion, stated in an unqualified fashion, was reached on the basis of a single-mode representation of the elastic displacements. It clearly cannot be sustained and is easily refuted by a more rigorous analysis. Nevertheless, the conclusion was never challenged and many investigators accepted it as fact. For example, Barbera and Likins[10] state that one must use rotating-appendage modes for a dynamic analysis and that nonrotating-cantilever modes are not applicable. In a later paper, Likins, Barbera, and Baddeley[11] refer to the practice of using nonrotating-appendage modes as a shortcut and cast doubt on its acceptability.

This paper shows that the system dynamic characteristics are not really affected by the type of functions used, provided that the set of functions is complete and a sufficiently large number of functions is taken. The class of functions used may have an influence only if severe truncation is involved. Even in such cases, surprisingly accurate natural frequencies can be obtained with only a handful of admissible functions, in many cases only one function per elastic displacement. These conclusions are the exact opposite of those reached by Rakowski and Renard.[9] It should be pointed out that a proper dynamical formulation already includes terms due to the rotation of the structure as a whole, which perhaps explains why the use of rotating-appendage eigenfunctions does not yield results much different from those obtained by use of any other set of admissible functions. Moreover, such eigenfunctions are not readily available, and they are likely more difficult to work with than a set of arbitrary admissible functions. Hence, the extra effort involved in the use of rotating-base eigenfunctions does not appear to be warranted. At the very least, the use of fixed-base eigenfunctions is fully justified.

Problem Formulation

Let us consider a rigid body with p flexible appendages rotating in space. The interest lies in a gyroscopic system, so that we shall consider the case in which the body is perturbed from uniform rotation. For the purpose of developing the theory, let us assume that the uniform rotation corresponds to the gravity-gradient configuration of a spacecraft, in which the mass center of the body moves in a circular orbit and the angular velocity of the body about its mass center is equal to the orbital angular velocity.

From Ref. 12, we conclude that the kinetic energy can be written in the matrix form

$$T=\tfrac{1}{2}\omega^T J\omega+\omega^T h+\tfrac{1}{2}\sum_{i=1}^{p}\int_{m_i}\dot{u}_i^T\dot{u}_i dm_i \tag{1}$$

where ω is the angular velocity vector of a convenient reference frame xyz, referred to as the global frame; h is the angular momentum vector due to elastic velocities alone; m_i is the mass of appendage i; $\dot{u}_i$ is the vector of elastic velocities relative to the rotating frame; and J is the inertia matrix of the entire body in deformed state. The expression for J is

$$J=\sum_{i=0}^{p}L_i^T J_i L_i \tag{2}$$

where J_0 is the inertia matrix of the entire body in undeformed state about the global body axes; $J_i\,(i=1,2,\ldots,p)$ is the change in the inertia matrix of the elastic appendage i about a set of local axes $x_i y_i z_i$; and L_i is the constant matrix of direction cosines between the local and global axes. Also from Ref. 12 we can write the gravitational potential energy in the form

$$V_G=-\tfrac{1}{2}\Omega^2 tr(J)+(3/2)\Omega^2\ell_a^T J\ell_a \tag{3}$$

where Ω is the orbital velocity and ℓ_a is the vector of direction cosines between the local vertical through the spacecraft mass center and the global frame. Note that ℓ_a depends on the angles $\theta_j\,(j=1,2,3)$, defining the orientation of the global axes with respect to the orbital axes abc, where a coincides with the local vertical, c is normal to the orbit plane, and b is oriented so as to form an orthogonal right-hand system. Moreover, let us assume that the elastic potential energy can be written in the general form

$$V_{EL}=\tfrac{1}{2}\sum_{i=1}^{p}\int_{D_i}(u_i'^T P_i u_i'+u_i''^T B_i u_i'')dD_i \tag{4}$$

where P_i and B_i are diagonal matrices of axial forces and flexural stiffnesses, respectively, and D_i represents the domain of extension of appendage i. Primes denote differentiations with respect to spatial variables.

The body angular velocity vector ω can be expressed in terms of the angular velocity Ω of the orbital axes and the angular velocity ω_l of the body relative to the orbital axes as follows:

$$\omega=\Omega\ell_c+\omega_l \tag{5}$$

where ℓ_c depends on the angular coordinates θ_j and ω_l depends on θ_j and $\dot{\theta}_j\,(j=1,2,3)$. Inserting Eq. (5) into Eq. (1), the kinetic energy can be written as

$$T=T_2+T_1+T_0 \tag{6}$$

where

$$T_2=\tfrac{1}{2}\omega_l^T J\omega_l+\omega_l^T h+\tfrac{1}{2}\sum_{i=1}^{p}\int_{m_i}\dot{u}_i^T\dot{u}_i dm_i \tag{7}$$

is quadratic in the generalized velocities

$$T_1=\Omega\ell_c^T(J\omega_l+h) \tag{8}$$

is linear in the generalized velocities, and

$$T_0=\tfrac{1}{2}\Omega^2\ell_c^T J\ell_c \tag{9}$$

is free of generalized velocities.

System Discretization and Linearization

For the sake of discussing the discretization problem, it will prove convenient to define certain classes of functions.[13] To this end, let us concentrate our attention on a given appendage in the form of an elastic continuum, and assume that it is possible to formulate an eigenvalue problem for the appendage, where the problem consists of a differential equation in the spatial variables and associated boundary conditions. The boundary conditions are of two types, geometric and natural. The first guarantees that the geometry is not violated at appropriate boundary points, and the second ensures force and moment balance at appropriate boundary points. The boundary conditions at a certain point can be of one type or the other, or a mixture of the two. The differential equation contains a given parameter, and at times the same parameter is contained in one or more of the boundary conditions. The eigenvalue problem consists of determining the set of values for the parameter and the set of associated functions that satisfy both the differential equations and all of the boundary conditions; these sets are denumerably infinite, and are known as *eigenvalues and eigenfunctions*, respectively. For self-adjoint systems, the eigenfunctions can be shown to be orthogonal. Quite often, however, it is not possible to solve the eigenvalue problem in closed form, in which case one may be content with an approximate solution. In this regard, there are two classes of functions of particular interest. One class encompasses functions satisfying all of the boundary conditions of the problem, but not necessarily the differential equation. This is the class of *comparison functions.* The second class encompasses functions satisfying the geometric boundary conditions alone, and not necessarily the differential equation nor the natural boundary conditions. These functions are known as *admissible functions.* Both the comparison and the admissible functions must satisfy certain differentiability requirements, but the requirements placed on admissible functions are less restrictive than those placed on comparison functions. Quite often, however, differentiability presents no problem, so that this fact is of no particular consequence. Clearly, the eigenfunctions represent a subset of the comparison functions, and the comparison functions represent a subset of the admissible functions. An important problem in structural dynamics, whether a structure is rotating or not, is the selection of functions to be used for the discretization process. The object is to use functions that are as simple as possible, and yet capable of yielding satisfactory results. In this regard, the admissible functions appear as a highly desirable class of functions.

Next, let us discretize the complete system. It is shown in Ref. 13 (Sec. 6-4) that if discretization is performed via the kinetic and potential energy, as opposed to the system partial differential equations of motion, then it is sufficient to use admissible functions to discretize a nongyroscopic system. Following the reasoning of Ref. 13, the same conclusion can be reached also for gyroscopic systems. Hence, let us assume that the elastic displacements can be represented as a superposition of space-dependent admissible functions multiplied by time-dependent generalized coordinates

$$u_i = \Phi_i \eta_i \tag{10}$$

where, in general, Φ_i is a $3\times 3n$ matrix of admissible functions and η_i is a $3n$-vector of generalized displacements. This permits us to eliminate the spatial dependence, so that

$$\sum_{i=1}^{p} \int_{m_i} \dot{u}_i^T \dot{u}_i dm_i = \dot{\eta}^T m_{\eta\eta} \dot{\eta} \tag{11}$$

where

$$m_{\eta\eta} = \sum_{i=1}^{p} \int_{m_i} \Phi_i^T \Phi_i dm_i \tag{12}$$

is a block-diagonal symmetric matrix and

$$\sum_{i=1}^{p} \int_{D_i} (u_i'^T P_i u_i' + u_i''^T B_i u_i'') dD_i = \eta^T k_{E\eta\eta} \eta \tag{13}$$

where

$$k_{E\eta\eta} = \sum_{i=1}^{p} \int_{D_i} (\Phi_i'^T P_i \Phi_i' + \Phi_i''^T B_i \Phi_i'') dD_i \tag{14}$$

also is a block-diagonal matrix. Quite often it is convenient to select the matrix Φ_i of admissible functions so that its rows constitute orthonormal sets, in which case $m_{\eta\eta}$ is equal to the unit matrix 1. We must recognize that J and h also involve spatial dependence. By use of the same transformation (10), we can eliminate the spatial dependence in J and h, so that we have $J = J(\eta)$ and $h = h(\eta, \dot{\eta})$. Hence, the Lagrangian can be expressed in the general functional form

$$L = T_2 + T_1 + T_0 - V_G - V_{EL} = L(\theta, \dot{\theta}, \eta, \dot{\eta}) \tag{15}$$

The equations of motion can be written in the form of Lagrange's equations

$$\begin{aligned} (d/dt)(\partial L/\partial \dot{\theta}) - \partial L/\partial \theta = 0 \\ (d/dt)(\partial L/\partial \dot{\eta}) - \partial L/\partial \eta = 0 \end{aligned} \tag{16}$$

where the partial derivative of L with respect to the vector θ is to be interpreted as a vector whose components are the partial derivatives of L with respect to the components of θ, etc.

For arbitrarily large angles θ_j, Eqs. (16) are nonlinear. Our interest, however, lies in small oscillations of the system about the eqilibrium corresponding to the steady rotation of the undeformed body with the angular velocity Ω, which implies the trivial equilibrium $\theta = \eta = \dot{\theta} = \dot{\eta} = 0$. Hence, the object is to linearize the system about the trivial equilibrium. The case in which the equilibrium is nontrivial can be reduced to the trivial case by a given coordinate transformation.[14] Linearization is tantamount to retaining only the quadratic terms in the generalized coordinates and velocities in the Lagrangian. The linearized Lagrangian can be written in the general form

$$\begin{aligned} L = {} & \tfrac{1}{2}\dot{\theta}^T m_{\theta\theta}\dot{\theta} + \dot{\theta}^T m_{\theta\eta}\dot{\eta} + \tfrac{1}{2}\dot{\eta}^T m_{\eta\eta}\dot{\eta} + \theta^T f_{\theta\theta}\dot{\theta} \\ & + \theta^T f_{\theta\eta}\dot{\eta} + \eta^T f_{\eta\theta}\dot{\theta} + \eta^T f_{\eta\eta}\dot{\eta} - \tfrac{1}{2}\theta^T k_{\theta\theta}\theta \\ & - \theta^T k_{\theta\eta}\eta - \tfrac{1}{2}\eta^T k_{\eta\eta}\eta \end{aligned} \tag{17}$$

where the matrices of the coefficients are

$$\left.\begin{aligned} m_{\theta\theta} &= \left[\frac{\partial^2 L}{\partial\dot{\theta}_i \partial\dot{\theta}_j}\right] = \left[\frac{\partial^2 T_2}{\partial\dot{\theta}_i \partial\dot{\theta}_j}\right] \\ f_{\theta\theta} &= \left[\frac{\partial^2 L}{\partial\theta_i \partial\dot{\theta}_j}\right] = \left[\frac{\partial^2 T_1}{\partial\theta_i \partial\dot{\theta}_j}\right] \\ k_{\theta\theta} &= -\left[\frac{\partial^2 L}{\partial\theta_i \partial\theta_j}\right] = \left[\frac{\partial^2 (V_G - T_0)}{\partial\theta_i \partial\theta_j}\right] \end{aligned}\right\} \quad i,j = 1,2,3$$

$$\left.\begin{aligned} m_{\theta\eta} &= \left[\frac{\partial^2 L}{\partial\dot{\theta}_i \dot{\eta}_j}\right] = \left[\frac{\partial^2 T_2}{\partial\dot{\theta}_i \partial\dot{\eta}_j}\right] \\ f_{\theta\eta} &= \left[\frac{\partial^2 L}{\partial\theta_i \partial\dot{\eta}_j}\right] = \left[\frac{\partial^2 T_1}{\partial\theta_i \partial\dot{\eta}_j}\right] \\ f_{\eta\theta} &= \left[\frac{\partial^2 L}{\partial\dot{\theta}_i \partial\eta_j}\right] = \left[\frac{\partial^2 T_1}{\partial\dot{\theta}_i \partial\eta_j}\right] \\ k_{\theta\eta} &= -\left[\frac{\partial^2 L}{\partial\theta_i \partial\eta_j}\right] = \left[\frac{\partial^2 (V_G - T_0)}{\partial\theta_i \partial\eta_j}\right] \end{aligned}\right\} \quad \begin{aligned} & i = 1,2,3; \\ & j = 1,2,\ldots,3n \end{aligned}$$

$$\left.\begin{aligned} m_{\eta\eta} &= \left[\frac{\partial^2 L}{\partial\dot{\eta}_i \partial\dot{\eta}_j}\right] = \left[\frac{\partial^2 T_2}{\partial\dot{\eta}_i \partial\dot{\eta}_j}\right] \\ f_{\eta\eta} &= \left[\frac{\partial^2 L}{\partial\eta_i \partial\dot{\eta}_j}\right] = \left[\frac{\partial^2 T_1}{\partial\eta_i \partial\dot{\eta}_j}\right] \\ k_{\eta\eta} &= -\left[\frac{\partial^2 L}{\partial\eta_i \partial\eta_j}\right] = \left[\frac{\partial^2 (V_G - T_0)}{\partial\eta_i \partial\eta_j}\right] + k_{E\eta\eta} \end{aligned}\right\} \quad i,j = 1,2,\ldots,3n \tag{18}$$

in which all of the partial derivatives are evaluated at equilibrium.

Inserting Eq. (17) into Eqs. (16), Lagrange's equations of motion can be written in the matrix form

$$m\ddot{q}+g\dot{q}+hq=0 \tag{19}$$

where

$$q=[\theta^T \mid \eta^T]^T \tag{20}$$

is an N-vector, including the rotational motion and the elastic generalized coordinates, $N=3(n+1)$, and

$$m=\left[\begin{array}{c|c} m_{\theta\theta} & m_{\theta\eta} \\ \hline m_{\theta\eta}^T & m_{\eta\eta}\end{array}\right], \qquad g=\left[\begin{array}{c|c} g_{\theta\theta} & g_{\theta\eta} \\ \hline -g_{\theta\eta}^T & g_{\eta\eta}\end{array}\right]$$
$$k=\left[\begin{array}{c|c} k_{\theta\theta} & k_{\theta\eta} \\ \hline k_{\theta\eta}^T & k_{\eta\eta}\end{array}\right] \tag{21}$$

where, in the second matrix,

$$g_{\theta\theta}=f_{\theta\theta}^T-f_{\theta\theta} \qquad g_{\eta\eta}=f_{\eta\eta}^T-f_{\eta\eta}$$
$$g_{\theta\eta}=f_{\eta\theta}^T-f_{\theta\eta} \qquad g_{\eta\theta}=f_{\theta\eta}^T-f_{\eta\theta}=-g_{\theta\eta}^T \tag{22}$$

It is easy to see that m and k are symmetric matrices and g is a skew symmetric matrix. Equation (19) represents a typical linear gyroscopic system, and the solution of the associated eigenvalue problem can be obtained by the method of Ref. 2.

The Stationarity Property of the Eigenvalue Problem

The solution of the eigenvalue problem associated with Eq. (19) can be treated more conveniently by working with the state vector instead of the configuration vector. Hence, introducing the $2N$-state vector

$$x(t)=\left[\begin{array}{c}\dot{q}(t)\\ \hline q(t)\end{array}\right] \tag{23}$$

and the $2N\times 2N$ matrices

$$I=\left[\begin{array}{c|c} m & 0 \\ \hline 0 & k\end{array}\right], \qquad G=\left[\begin{array}{c|c} g & k \\ \hline -k & 0\end{array}\right] \tag{24}$$

the eigenvalue problem associated with system (19) can be written in the real symmetric form[2]

$$\omega_r^2 Iy_r=Ky_r, \quad \omega_r^2 Iz_r=Kz_r, \quad r=1,2,\ldots,N \tag{25}$$

where ω_r are the natural frequencies of oscillation of the system and y_r and z_r are the real eigenvectors, corresponding to the real and imaginary parts of x_r. Moreover,

$$K=G^TI^{-1}G \tag{26}$$

is easily recognized as being a symmetric matrix. The eigenvalues ω_r^2 have multiplicity two, and the eigenvector pair y_r and z_r belongs to ω_r^2. If I is positive definite, then K is positive definite, and the set of eigenvectors are orthogonal with respect to I. They can be normalized conveniently so as to form an orthonormal set satisfying

$$y_r^TIy_s=z_r^TIz_s=\delta_{rs}, \quad y_r^TIz_s=0 \tag{27}$$

It follows immediately from Eqs. (25) and (27) that

$$y_r^TKy_s=z_r^TKz_s=\omega_r^2\delta_{rs}, \quad y_r^TKz_s=0 \tag{28}$$

Next let us consider the expression

$$R(v)=v^TKv/v^TIv \tag{29}$$

where v is any arbitrary real $2N$-vector, and $R(v)$ is referred to as *Rayleigh's quotient for gyroscopic systems.* Rayleigh's quotient has been used widely in conjunction with nonrotating elastic systems, defined by symmetric matrices and vectors in the configuration space.[13] The stationarity properties of Rayleigh's quotient have proved very useful in obtaining eigenvalue estimates for such systems; they are not limited to real symmetric matrices and, indeed, they exist also for Hermitian matrices.[15] Note that the modal matrix of a Hermitian matrix is unitary, and hence complex. In this paper we wish to establish the stationarity of Rayleigh's quotient for gyroscopic systems Eq. (29), working only with real state vectors.

According to the expansion theorem of Ref. 2, any arbitrary state vector can be expressed as a superposition of the eigenvectors y_r and z_r in the form

$$v=\sum_{r=1}^{n}(a_ry_r+b_rz_r) \tag{30}$$

Inserting Eq. (30) into Eq. (29) and considering Eqs. (27) and (28), we obtain

$$R=\frac{\sum_{r=1}^{N}\sum_{s=1}^{N}(a_ry_r^T+b_rz_r^T)K(a_sy_s+b_sz_s)}{\sum_{r=1}^{N}\sum_{s=1}^{N}(a_ry_r^T+b_rz_r^T)I(a_sy_s+b_sz_s)}=\frac{\sum_{r=1}^{N}\omega_r^2(a_r^2+b_r^2)}{\sum_{r=1}^{N}(a_r^2+b_r^2)} \tag{31}$$

Next, let us assume that the trial vector v resembles closely one of the eigenvectors, say y_t. The implication is that the coefficient a_t in expansion (30) is larger than the remaining coefficients by at least one order of magnitude. Mathematically, this can be expressed by

$$a_r/a_t=\epsilon_r, \quad r\neq t, \quad b_r/a_t=\epsilon_{N+r} \tag{32}$$

where ϵ_r and ϵ_{N+r} are small quantities. Dividing the numerator and denominator on the extreme right of Eq. (31) by a_t^2, and using Eqs. (32), we can write

$$R=\frac{\omega_t^2+\sum_{\substack{r=1\\ r\neq t}}^{N}\omega_r^2\epsilon_r^2+\sum_{r=1}^{N}\omega_r^2\epsilon_{N+r}^2}{1+\sum_{\substack{r=1\\ r\neq t}}^{N}\epsilon_r^2+\sum_{r=1}^{N}\epsilon_{N+r}^2}$$
$$\simeq\omega_t^2+\sum_{\substack{r=1\\ r\neq t}}^{N}(\omega_r^2-\omega_t^2)\epsilon_r^2+\sum_{r=1}^{N}(\omega_r^2-\omega_t^2)\epsilon_{N+r}^2=\omega_t^2+O(\epsilon^2) \tag{33}$$

where $O(\epsilon^2)$ denotes a quantity of second order in ϵ. The implication of Eq. (34) is that *if the trial vector v differs from the eigenvector y_t by a quantity of first order in ϵ, then Rayleigh's quotient differs from the eigenvalue ω_t^2 by a quantity of second order in ϵ*. Hence, *Rayleigh's quotient has a stationary value in the neighborhood of an eigenvalue.* Letting $t=1$, Eq. (33) reduces to

$$R\simeq\omega_1^2+\sum_{r=2}^{N}(\omega_r^2-\omega_1^2)\epsilon_r^2+\sum_{r=1}^{N}(\omega_r^2-\omega_1^2)\epsilon_{N+r}^2\geq\omega_1^2 \tag{34}$$

or, *Rayleigh's quotient provides an upper bound for the lowest eigenvalue.* It also can be shown that Rayleigh's quotient provides an upper bound for ω_r^2 $(2\leq r\leq N)$ with respect to all $2N$-vectors belonging to the subset of vectors orthogonal to the first $2(r-1)$ eigenvectors.

All of the above properties are well known for nonrotating linear systems (see, for example, Ref. 13), but the existence of these properties for linear gyroscopic systems provides a powerful tool for the modal analysis of rotating structures. The physical implication of the properties proved in the preceding is that the solution of the eigenvalue problem is relatively insensitive to the set of admissible functions used for the system discretization, provided that the set is complete. The calculated eigenvalues approach the true eigenvalues asymptotically as the degree of freedom of the simulation is increased. In many cases, however, surprising accuracy can be achieved with only a limited number of admissible functions.

Appendage Eigenvalue Problem

The question of discretization of flexible appendages and that of series truncation, to limit the number of degrees of freedom of the simulation, leads to another question, namely, the desirability of using the appendage eigenfunctions as admissible functions, particularly if the appendages are subjected to a centrifugal force field. Quite often, appendage eigenvalue problems do not involve Coriolis effects, in which case a Rayleigh-Ritz procedure[13] leads to the eigenvalue problem

$$\Lambda_r^2 m_{\eta\eta}\eta_r = k_{\eta\eta}\eta_r \qquad r=1,2,\ldots,3n \tag{35}$$

where $m_{\eta\eta}$ and $k_{\eta\eta}$ are the matrices defined in Eqs. (18). Note that Eq. (35) may involve p independent eigenvalue problems, one for each appendage. If the eigenvectors are normalized so as to satisfy $\eta_r^T m_{\eta\eta}\eta_s = \delta_{rs}$, then the coordinate transformation

$$\eta = U\xi \tag{36}$$

where $U=[\eta_1\eta_2,\ldots,\eta_{3n}]$ is the orthonormal modal matrix of the appendages, diagonalizes $m_{\eta\eta}$ and $k_{\eta\eta}$ simultaneously. More specifically, U is such that

$$U^T m_{\eta\eta} U = 1 \qquad U^T k_{\eta\eta} U = \Lambda^2 \tag{37}$$

where 1 is the unit matrix of order $3n$, and Λ^2 is a diagonal matrix of the eigenvalues Λ_r^2.

Introducing Eq. (36) into Eq. (17), and considering Eqs. (37), we obtain the Lagrangian

$$L = \tfrac{1}{2}\dot{\theta}^T m_{\theta\theta}\dot{\theta} + \dot{\theta}^T m_{\theta\xi}\dot{\xi} + \tfrac{1}{2}\dot{\xi}^T\dot{\xi} + \theta^T f_{\theta\theta}\dot{\theta}$$
$$+\theta^T f_{\theta\xi}\dot{\xi} + \xi^T f_{\xi\theta}\dot{\theta} + \xi^T f_{\xi\xi}\dot{\xi} - \tfrac{1}{2}\theta^T k_{\theta\theta}\theta - \theta^T k_{\theta\xi}\xi - \tfrac{1}{2}\xi^T\Lambda^2\xi \tag{38}$$

where

$$m_{\theta\xi} = m_{\theta\eta}U, \quad k_{\theta\xi} = k_{\theta\eta}U$$
$$f_{\theta\xi} = f_{\theta\eta}U, \quad f_{\xi\theta} = U^T f_{\eta\theta}, \quad f_{\xi\xi} = U^T f_{\eta\eta}U \tag{39}$$

As a result, Lagrange's equations become

$$m^*\ddot{q}^* + g^*\dot{q}^* + k^*q^* = 0 \tag{40}$$

where

$$q^* = [\theta^T \mid \xi^T]^T \tag{41}$$

and

$$m^* = \left[\begin{array}{c|c} m_{\theta\theta} & m_{\theta\xi} \\ \hline m_{\theta\xi}^T & 1 \end{array}\right], \qquad g^* = \left[\begin{array}{c|c} g_{\theta\theta} & g_{\theta\xi} \\ \hline -g_{\theta\xi}^T & g_{\xi\xi} \end{array}\right]$$
$$k^* = \left[\begin{array}{c|c} k_{\theta\theta} & k_{\theta\xi} \\ \hline k_{\theta\xi}^T & \Lambda^2 \end{array}\right] \tag{42}$$

where $g_{\theta\xi}$ and $g_{\xi\xi}$ are given by expressions analogous to Eqs. (22).

Comparing Eqs. (21) and (42), it is clear that no particular advantage accrues from using the eigenfunctions of the uniformly rotating appendages as admissible functions, because the equations of motion remain coupled through the rotational coordinates. Moreover, the simplification of some of the matrix coefficients is not as significant as one may be tempted to think. On the other hand, the expressions for the eigenfunctions are likely to be more complicated and difficult to work with than those for an arbitrary set of admissible functions. It should be pointed out that there is a large degree of latitude in choosing admissible functions, since they are in great abundance, although quite often a convenient set of admissible functions is the set of eigenfunctions of the corresponding fixed-base elastic member. This latter set also is orthogonal with respect to $m_{\eta\eta}$, and can be normalized so that $m_{\xi\xi}=1$, in which case the only real simplification realized in using the eigenfunctions of the rotating appendage as admissible functions is in the matrix k^* as opposed to k. Hence, the appendage eigenvalue problem may present an interest when a structure is constrained to rotate uniformly about a fixed axis in space, in which case there are no rotational degrees of freedom involved. However, the interest is drastically reduced when the structure can nutate, such as in the case of a structure rotating freely in space.

Illustrative Example

Let us consider a spacecraft consisting of a rigid body with a pair of identical booms, as shown in Fig. 1. The booms are aligned with the local vertical when in equilibrium. The axial deformation of the booms is assumed to be negligibly small, so that only the bending displacements v and w are present. Moreover, the elastic deformations are assumed to be antisymmetric, $v(x,t) = -v(-x,t)$, $w(x,t) = -w(-x,t)$. The mass density of each boom is $\tfrac{1}{2}\rho$, but the fact that the booms are identical permits us to regard the two booms as if they were a single boom of mass density ρ. Assuming that the dimensions of the center body are small as compared to the length L of the booms, the inertia matrix of the spacecraft in deformed state is

$$J = \begin{bmatrix} A + \int_0^L \rho(v^2+w^2)\,dx & -\int_0^L \rho xv\,dx & -\int_0^L \rho xw\,dx \\ -\int_0^L \rho xv\,dx & B + \int_0^L \rho w^2\,dx & -\int_0^L \rho vw\,dx \\ -\int_0^L \rho xw\,dx & -\int_0^L \rho vw\,dx & C + \int_0^L \rho v^2\,dx \end{bmatrix} \tag{43}$$

where A, B, and C are the principle moments of inertia of the spacecraft about x,y, and z, respectively. Note that in this case the global and local axes coincide. Assuming that the body axes xyz are obtained from abc by means of the rotations θ_2 about y, θ_1 about x, and θ_3 about z, the vectors of direction cosines are

$$\ell_c = \begin{bmatrix} s\theta_1 c\theta_2 s\theta_3 - s\theta_2 c\theta_3 \\ s\theta_1 c\theta_2 c\theta_3 + s\theta_2 s\theta_3 \\ c\theta_1 c\theta_2 \end{bmatrix}, \quad \ell_a = \begin{bmatrix} s\theta_1 s\theta_2 s\theta_3 + c\theta_2 c\theta_3 \\ s\theta_1 s\theta_2 c\theta_3 - c\theta_2 s\theta_3 \\ c\theta_1 s\theta_2 \end{bmatrix} \tag{44}$$

where $s\theta_j = \sin\theta_j$, $c\theta_j = \cos\theta_j$ $(j=1,2,3)$. The angular velocity vector of axes xyz relative to abc is

$$\omega_L = \begin{bmatrix} \dot{\theta}_1 c\theta_3 + \dot{\theta}_2 c\theta_1 s\theta_3 \\ -\dot{\theta}_1 s\theta_3 + \dot{\theta}_2 c\theta_1 c\theta_3 \\ -\dot{\theta}_2 s\theta_1 + \dot{\theta}_3 \end{bmatrix} \tag{45}$$

and the angular momentum vector due to elastic velocities alone is

$$h=\begin{bmatrix} \int_0^L \rho(v\dot{w}-\dot{v}w)\,dx \\ -\int_0^L \rho x\dot{w}\,dx \\ \int_0^L \rho x\dot{v}\,dx \end{bmatrix} \tag{46}$$

In the following, we shall consider the fact that the axial deformation is zero, so that the associated generalized coordinates will be ignored. As a result, we have only $2n$ elastic degrees of freedom, corresponding to v and w, instead of $3n$. The axial force is due to the centrifugal force and the differential gravity. The combined effect yields the axial force matrix

$$P=(3/2)\rho\Omega^2(L^2-x^2)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{47}$$

Similarly, the flexural stiffness matrix can be written as

$$B=EI\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{48}$$

Denoting the admissible functions for v by $\phi_1, \phi_2, \ldots, \phi_n$, and those for w by $\psi_1, \psi_2, \ldots, \psi_n$, the matrix of admissible functions is

$$\Phi=\begin{bmatrix} \phi_1 & \phi_2 & ---- & 0 & 0 & 0 & ---- & 0 \\ 0 & 0 & ---- & 0 & \psi_1 & \psi_2 & ---- & \psi_n \end{bmatrix} \tag{49}$$

so that, introducing the notation

$$a_i=\int_0^L \rho x\phi_i\,dx, \quad b_i=\int_0^L \rho x\psi_i\,dx, \quad i=1,2,\ldots,n \tag{50}$$

and using Eqs. (18) in conjunction with Eqs. (12, 14, and 22), we obtain the submatrices

$$m_{\theta\theta}=\begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}, \qquad m_{\eta\eta}=\left[\begin{array}{c|c} m_{\phi ij} & 0 \\ \hline 0 & m_{\psi ij} \end{array}\right]$$

$$m_{\theta\eta}=\begin{bmatrix} 0 & 0 & ---- & 0 & 0 & 0 & ---- & 0 \\ 0 & 0 & ---- & 0 & -b_1 & -b_2 & ---- & -b_n \\ a_1 & a_2 & ---- & a_n & 0 & 0 & ---- & 0 \end{bmatrix}$$

$$g_{\theta\theta}=\Omega\begin{bmatrix} 0 & -(A+B-C) & 0 \\ A+B-C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad g_{\theta\eta}=g_{\eta\eta}=0$$

$$k_{\theta\theta}=\Omega^2\begin{bmatrix} C-B & 0 & 0 \\ 0 & 4(C-A) & 0 \\ 0 & 0 & 3(B-A) \end{bmatrix} \tag{51}$$

$$k_{\theta\eta}=\Omega^2\begin{bmatrix} 0 & 0 & ---- & 0 & 0 & 0 & ---- & 0 \\ 0 & 0 & ---- & 0 & -4b_1 & -4b_2 & ---- & -4b_n \\ 3a_1 & 3a_2 & ---- & 3a_n & 0 & 0 & ---- & 0 \end{bmatrix}$$

$$k_{\eta\eta}=\Omega^2\left[\begin{array}{c|c} c_{ij} & 0 \\ \hline 0 & m_{\psi ij}+d_{ij} \end{array}\right]$$

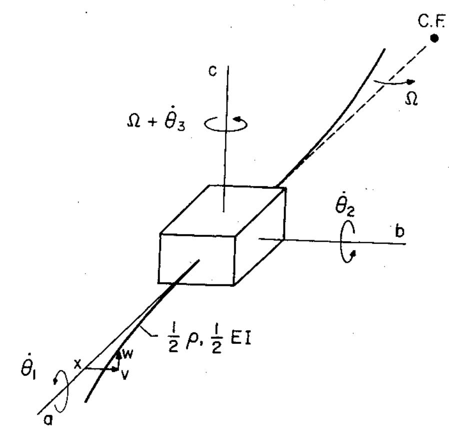

Fig. 1 Rotating flexible structure.

where

$$m_{\phi ij}=\int_0^L \rho\phi_i\phi_j\,\mathrm{d}x, \quad m_{\psi ij}=\int_0^L \rho\psi_i\psi_j\,\mathrm{d}x$$

$$c_{ij}=\int_0^L\left[\frac{3}{2}\rho(L^2-x^2)\phi_i'\phi_j'+\frac{EI}{\Omega^2}\phi_i''\phi_j''\right]\mathrm{d}x \tag{52}$$

$$d_{ij}=\int_0^L\left[\frac{3}{2}\rho(L^2-x^2)\psi_i'\psi_j'+\frac{EI}{\Omega^2}\psi_i''\psi_j''\right]\mathrm{d}x$$

$i,j=1,2,\ldots,n$

Inserting Eqs. (51) into Eqs. (21), we can form matrices (24), which in turn permits us to solve the eigenvalue problem (25) and (26). The problem has been solved four times, using the following parameters:

$$\begin{aligned} &A=90 \text{ slug-ft}^2, \quad B=28{,}880 \text{ slug-ft}^2 \\ &C=29{,}000 \text{ slug-ft}^2 \\ &\rho=4\times 10^{-4} \text{ slug-ft}^{-1}, \quad L=600 \text{ ft} \\ &EI=15 \text{ lb-ft}^2, \quad \Omega=5\times 10^{-4} \text{ rad s}^{-1} \end{aligned} \tag{53}$$

The first solution was obtained by using an expansion in terms of the admissible functions

$$\phi_r(x)=\psi_r(x)=(x/L)^{r+1}, \quad r=1,2,\ldots \tag{54}$$

The natural frequencies are displayed in column a of Tables 1-4 for $r=1, 2, 3, 4$, respectively. Note that the functions (54) satisfy only the geometric boundary conditions, namely, they are such that $\phi_r(0)=\psi_r(0)=\phi_r'(0)=\psi_r'(0)=0$, where primes denote derivatives with respect to x. The second solution is obtained by use of the comparison functions

$$\phi_r(x)=\psi_r(x)=1-\cos r\pi x/L+\tfrac{1}{2}(-1)^{r+1}(r\pi x/L)^2$$
$$r=1,2,\ldots \tag{55}$$

which satisfy all of the boundary conditions of the appendage, $\phi_r(0)=\psi_r(0)=\phi_r'(0)=\psi_r'(0)=\phi_r''(L)=\psi_r''(L)=\phi_r'''(L)=\psi_r'''(L)=0$. The corresponding natural frequencies are listed in column b of the same tables. The third solution was obtained by using, as comparison functions the fixed-base cantilever functions. The functions satisfy the differential equation

$$\phi_r''''-\beta_r^4\phi_r=0, \quad \beta_r^4=\Lambda_r^2\rho/EI, \quad r=1,2,\ldots \tag{56}$$

Table 1 Spacecraft natural frequencies: one-term series expansion

	a	b	c	d
$\omega_1 \times 10^3$	0.57726374	0.57723467	0.57722628	0.57720431
$\omega_2 \times 10^3$	0.86284666	0.86282457	0.86281847	0.86280416
$\omega_3 \times 10^2$	0.10005826	0.10005818	0.10005815	0.10005810
$\omega_4 \times 10^2$	0.26282522	0.27010909	0.27779727	0.34308837
$\omega_5 \times 10^2$	0.27599779	0.28977172	0.30061319	0.39315230

Table 2 Spacecraft natural frequencies: two-term series expansion

	a	b	c	d
$\omega_1 \times 10^3$	0.57722517	0.57721461	0.57720874	0.57720394
$\omega_2 \times 10^3$	0.86281847	0.86281171	0.86280756	0.86280414
$\omega_3 \times 10^2$	0.10005815	0.10005813	0.10005811	0.10005810
$\omega_4 \times 10^2$	0.26082199	0.25391632	0.25412733	0.25396679
$\omega_5 \times 10^2$	0.27543235	0.26453340	0.26461209	0.26452496
$\omega_6 \times 10^2$	0.61718134	0.52795685	0.54015777	0.56376815
$\omega_7 \times 10^2$	0.68891855	0.62929835	0.66614578	0.72454327

Table 3 Spacecraft natural frequencies: three-term series expansion

	a	b	c	d
$\omega_1 \times 10^3$	0.57720782	0.57720831	0.57720486	0.57720392
$\omega_2 \times 10^3$	0.86280687	0.86280723	0.86280480	0.86280414
$\omega_3 \times 10^2$	0.10005811	0.10005811	0.10005810	0.10005810
$\omega_4 \times 10^2$	0.25386242	0.25371693	0.25363059	0.25365627
$\omega_5 \times 10^2$	0.26442481	0.26445430	0.26423340	0.26424269
$\omega_6 \times 10^2$	0.53081440	0.51567968	0.51445116	0.51579775
$\omega_7 \times 10^2$	0.63665645	0.57241497	0.57130770	0.57308961
$\omega_8 \times 10$	0.13136250	0.08367958	0.08361849	0.08383451
$\omega_9 \times 10$	0.14451872	0.10132162	0.10483690	0.10632853

Table 4 Spacecraft natural frequencies: four-term series expansion

	a	b	c	d
$\omega_1 \times 10^3$	0.57720400	0.57720593	0.57720392	0.57720392
$\omega_2 \times 10^3$	0.86280421	0.86280557	0.86280414	0.86280414
$\omega_3 \times 10^2$	0.10005810	0.10005810	0.10005810	0.10005810
$\omega_4 \times 10^2$	0.25384627	0.25363624	0.25362895	0.25362895
$\omega_5 \times 10^2$	0.26442481	0.26424570	0.26421796	0.26421796
$\omega_6 \times 10^2$	0.51686910	0.51449864	0.51427631	0.51427632
$\omega_7 \times 10^2$	0.57605387	0.57224632	0.57126903	0.57126902
$\omega_8 \times 10$	0.08471890	0.08355646	0.08272759	0.08272759
$\omega_9 \times 10$	0.10665488	0.09827929	0.09813977	0.09813977
$\omega_{10} \times 10$	0.27825175	0.13465840	0.13314028	0.13314028
$\omega_{11} \times 10$	0.28789581	0.14798726	0.14899459	0.14899459

and all of the boundary conditions of the appendage, in which Λ_r are the natural frequencies of the fixed-base cantilever beam. From Ref. 13 (p. 163), we obtain

$$\phi_r(x) = (\sin\beta_r L - \sinh\beta_r L)(\sin\beta_r x - \sinh\beta_r x) + (\cos\beta_r L + \cosh\beta_r L)(\cos\beta_r x - \cosh\beta_r x) \quad r = 1,2,\ldots \tag{57}$$

The functions $\psi_r(x)$ are identical to $\phi_r(x)$. The natural frequencies corresponding to these comparison functions are shown in column c of Tables 1-4.

The fourth solution makes use of the rotating-base cantilever eigenfunctions. Since these eigenfunctions are not readily available, it is first necessary to solve the eigenvalue problem for the rotating appendage. We note that the appendage is subjected to both centrifugal and differential-gravity forces. As a result, the eigenvalue problems for ϕ_r and ψ_r are different from those in which only centrifugal forces are present. Indeed, it is easy to verify that ϕ_r must satisfy the eigenvalue problem

$$EI\phi_r'''' - [(3/2)\rho\Omega^2(L^2 - x^2)\phi_r']' = \rho\Lambda_{\phi r}^2\phi_r \tag{58}$$

whereas ψ_r must satisfy

$$EI\psi_r'''' - [(3/2)\rho\Omega^2(L^2 - x^2)\psi_r']' + \rho\Omega^2\psi_r = \rho\Lambda_{\psi r}^2\psi_r \tag{59}$$

Of course, both ϕ_r and ψ_r must satisfy all of the boundary conditions of the problem, which are the same as those for a fixed-base cantilever beam. The solutions of Eqs. (58) and (59) are obtained by the Rayleigh-Ritz method, using as a set of comparison functions the first four terms in (57). This yields two 4×4 eigenvalue problems, one for the in-plane displacements and the other for the out-of-plane. The solution consists of the in-plane and out-of-plane natural frequencies and modal vectors. The modal vectors are to be used to represent the solutions of Eqs. (58) and (59) as linear combinations of the comparison functions (57). For example, the solution ϕ_l of Eq. (58) is obtained by multiplying each element of the first eigenvector by the correspondingly indexed function in (57) and summing up. The spacecraft natural frequencies corresponding to series expansions in terms of ϕ_r and ψ_r thus obtained (i.e., ϕ_r and ψ_r that solve Eqs. (58) and (59), respectively) are displayed in column d of Tables 1-4. As a matter of possible interest, the "appendage natural frequencies" $\Lambda_{\phi r}$ and $\Lambda_{\psi r}$ are shown in Table 5.

Table 5 Natural frequencies of rotating flexible appendage (obtained by the Rayleigh-Ritz method)

r	$\Lambda_{\phi r}$	$\Lambda_{\psi r}$
1	$0.98233502 \times 10^{-3}$	$0.11022623 \times 10^{-2}$
2	$0.30366961 \times 10^{-2}$	$0.30775840 \times 10^{-2}$
3	$0.68959694 \times 10^{-2}$	$0.69140722 \times 10^{-2}$
4	$0.12588921 \times 10^{-1}$	$0.12598846 \times 10^{-1}$

From Tables 1-4, we observe that the class of functions used has virtually no effect on the three lowest natural frequencies of the spacecraft. As the number of terms in the series expansions increases, all of the calculated frequencies improve, tending to their true values asymptotically. Of course, the number of frequencies coincides with the number of degrees of freedom of the simulation, which in turn depends on the number of terms in the series. Significant differences appear only in the higher frequencies, which is to be expected. The results using four-term expansions are remarkable. There is less than 1% difference in the lowest seven frequencies, regardless of the class of functions used. We also note that columns c and d of Table 4 are virtually identical. This can be explained easily by the fact that these columns represent solutions of the same eigenvalue problem, but are obtained differently. In column d the results are obtained in two stages. In the first stage the rotating-appendage eigenvalue problem is solved for the first four modes, and in the second stage these four modes are used toward solution of the spacecraft eigenvalue problem. By contrast, in column c the spacecraft eigenvalue problem is solved in a single stage, so that explicit solution of the appendage eigenvalue problem is bypassed entirely. Identical results can be results can be expected any time the order of the appendage eigenvalue problem is equal to the number of functions used to discretize the appendage.

Conclusions

This paper brings the question of discretization of rotating structures into sharp focus. It raises and answers the question as to the type of functions to be used in discretization. The answer is provided by a stationarity principle for linear gyroscopic systems, developed and proved in this paper. The implication of the principle is that the Rayleigh-Ritz approach can be used for rotating structures in a manner entirely analogous to that for nonrotating structures. Indeed, the paper demonstrates that discretization by the assumed modes method, using a set of admissible functions, can yield results that are as satisfactory as those obtained using the eigenfunctions of the rotating appendage, provided the set of admissible functions is complete. The paper demonstrates clearly that the practice of using the eigenfunctions of the fixed-base appendage yields highly satisfactory results, so that the use of rotating-appendage eigenfunctions for discretization purposes is unnecessary.

The stationarity principle has important implications not only in a modal analysis for the response, but also in a stability analysis of flexible spacecraft. The theory presented in this paper should not only help develop efficient computational procedures, but also should go a long way toward improving the understanding of the behavior of linear gyroscopic systems.

References

[1]Meirovitch, L., *Elements of Vibration Analysis*, McGraw-Hill Book Co., New York, 1975.

[2]Meirovitch, L., "A New Method of Solution of the Eigenvalue Problem for Gyroscopic Systems," *AIAA Journal*, Vol. 12, Oct. 1974, pp. 1337-1342.

[3]Thomson, W. T. and Reiter, G. S., "Attitude Drift of Space Vehicles," *The Journal of Astronautical Sciences*, Vol. 7, 1970, pp. 29-34.

[4]Buckens, F., "On the Influence of the Elasticity of the Components in a Spinning Satellite on the Stability of its Motion," Proceedings of the XVI International Astronautical Congress, Athens, 1965, pp. 327-342.

[5]Meirovitch, L. and Nelson, H. D., "On the High-Spin Motion of a Satellite Containing Elastic Parts," *Journal of Spacecraft and Rockets*, Vol. 3, Nov. 1966, pp. 1597-1602.

[6]Ashley, H., "Observations on the Dynamic Behavior of Large Flexible Bodies in Orbit," *AIAA Journal*, Vol. 5, March 1967, pp. 460-469.

[7]Dokuchaev, L. V., "Plotting the Regions of Stable Rotation of a Space Vehicle with Elastic Rods," transl. from *Kosmicheskie Issledovanyia*, Vol. 7, No. 4, 1969, pp. 534-546.

[8]Vigneron, F. R., "Stability of a Freely Spinning Satellite of Crossed-Dipole Configuration," *Canadian Aeronautics and Space Institute Transactions,* Vol. 3, No. 1, 1970, pp. 8-19.

[9]Rakowski, J. E. and Renard, M. L., "A Study of the Nutational Behavior of a Flexible Spinning Satellite Using Natural Frequencies and Modes of the Rotating Structure," presented as Paper 70-1046 at the AIAA Astrodynamics Conference, Santa Barbara, Calif., 1970.

[10]Barbera, F. J. and Likins, P. W., "Liapunov Stability Analysis of Spinning Spacecraft," *AIAA Journal*, Vol. 11, April 1973, pp. 457-466.

[11]Likins, P. W., Barbera, F. J., and Baddeley, V., "Mathematical Modeling of Spinning Elastic Bodies for Modal Analysis," *AIAA Journal*, Vol. 11, Sept. 1973, pp. 1251-1258.

[12]Meirovitch, L., "Liapunov Stability Analysis of Hybrid Dynamical Systems with Multi-Elastic Domains," *International Journal of Non-Linear Mechanics*, Vol. 7, 1972, pp. 425-443.

[13]Meirovitch, L., *Analytical Methods in Vibrations*, The Macmillan Co., New York, 1967.

[14]Meirovitch, L. and Juang, J. N., "On the Natural Modes of Oscillation of Rotating Flexible Structures about Nontrivial Equilibrium," *Journal of Spacecraft and Rockets*, Vol. 13, Jan. 1976, pp. 37-44.

[15]Franklin, J. N., *Matrix Theory*, Prentice-Hall, Englewood Cliffs, N. J., 1968.